$J_0(N)$ mod N has multiplicative reduction

Introduction:

Let N be a prime (we may assume $N > 7$)

Let $X_0(N)$ be the (compactified) modular curve for $\Gamma_0(N)$.

Let $J_0(N)$ be the Jacobian of $X_0(N)$

Recall:

$X_0(N)$ has a smooth & proper model over $\mathbb{Z}[\frac{1}{N}]$.

$\Downarrow$

$J_0(N)$ has a smooth & proper model over $\mathbb{Z}[\frac{1}{N}]$

$\Downarrow$

Neron model (over $\mathbb{Z}$) of $J_0(N)$ is an abelian scheme over $\mathbb{Z}[\frac{1}{N}]$.

$\Downarrow$

Qn: What are the fibers of this Neron model at N?

Why is this Qn. important?

We have the following result.

Thm. Suppose A is an abelian variety (over $\mathbb{Q}$) with a map $f: X_0(N) \to A$ such that

i) A has completely good reduction away from N.

ii) A has toric (or multiplicative) reduction at N.

iii) The JH-components of $A[p](\bar{\mathbb{Q}})$ are trivial or cyclotomic for some prime $p \neq N$.

iv) $f(0) \neq f(\infty)$ (separates cusps)

Then no Elliptic curve over $\mathbb{Q}$ has a rational point of order N.

- In this lecture we will show ii) can be guaranteed for all quotients A of $J_0(N)$ by showing it for $J_0(N)$.
- Note that

$$
\begin{tikzcd}
X_0(N) \arrow[r, "f"] \arrow[d] & A \\
J_0(N) \arrow[ur, dashed] &
\end{tikzcd}
$$

$J_0(N) \rightarrow$ Albanese of $X_0(N)$ = Jacobian of $X_0(N)$

Strategy of the Proof:

1) Relate the reduction of $J_0(N)$ mod N with the Pic^0 of the reduction of the minimal regular model of $X_0(N)$ at N

$$
\begin{tikzcd}
X_0(N)_{\mathbb{Q}} \arrow[r] \arrow[d] & J_0(N)_{\mathbb{Q}} \arrow[d] \\
\overline{X_0(N)}_{\mathbb{Z}} \arrow[d, "\text{reduce mod } N"] & \mathbb{J}_0(N)_{\mathbb{Z}} \arrow[d, "\text{reduce mod } N"] \\
\overline{X_0(N)}_{\mathbb{F}_N} \arrow[dr, "\mathrm{Pic}^0"'] & \mathbb{J}_0(N)_{\mathbb{F}_N} \arrow[dl, "(\ )^0"] \\
& \mathrm{Pic}^0(\overline{X_0(N)}) = (\mathbb{J}_0(N)_{\mathbb{F}_N})^0
\end{tikzcd}
$$

Raynaud's Theorem.

minimal regular model $\rightarrow \overline{X_0(N)}_{\mathbb{Z}}$

$\mathbb{J}_0(N)_{\mathbb{Z}}$ $\leftarrow$ Neron Model of $J_0(N)_{\mathbb{Q}}$

not smooth but mild singularities $\rightarrow \overline{X_0(N)}_{\mathbb{F}_N}$

$\mathbb{J}_0(N)_{\mathbb{F}_N}$ $\leftarrow$ not proper but smooth (over $\mathbb{F}_N$)

$(\ )^0 \leftarrow$ identity component

We shall discuss the precise statement in a while.

2) Get hold of singularities of $\overline{X_0}(N)_{\mathbb{F}_N} \to$ Basically be a bunch of $\mathbb{P}^1$'s intersecting transversally.

As Snowden does it, we shall spend most of our time with 2) taking 1) for granted.

Completely Toric reduction for Quotients:

Lemma: Let R be a Dedekind Domain with $Q(R)=K$. Let A & B be Abelian varieties over K s.t $A \twoheadrightarrow B$. Then A has a completely toric reduction at a closed point of R $\Rightarrow B$ has " " " " " the closed pt.

Rmk: $A/_K$ is said to have completely toric reduction at k if $(\mathcal{A}_k)^{\circ}$ ← connected component of identity $\xrightarrow{\sim}$ Torus (not necessarily split) over k.

Pf. Let $f: A \to B$ be the quotient map.

$\Rightarrow \exists\ g: B \to A$ s.t $g \circ f = n_A$.

By NMP f & g extend (uniquely) to $F: \mathcal{A} \to \mathcal{B}$
$G: \mathcal{B} \to \mathcal{A}$.

Moreover $G \circ F$ is the unique extension of $g \circ f$. Since $\mathcal{A}$ is also a comm. gp. scheme $n_{\mathcal{A}}$ makes sense & by uniqueness of NMP $G \circ F = n_{\mathcal{A}} \Rightarrow B_k$ is a quotient of A_k

$\Rightarrow B_k^\circ$ is a quotient of a finite index subgp of A_k.
with A_k° a torus

$\Rightarrow B_k^\circ$ is a torus (can check this over k_{sep}).

Raynaud's Theorem:

Let $f: X \to S$ proper flat map of (locally) Noetherian schemes.

$\mathrm{Pic}_{X/S} \to$ fppf sheafification of the presheaf of gps.

$$S' \mapsto \mathrm{Pic}(X_{S'}) \text{ in the big fppf site over } S.$$

Fact: When S is a field $\mathrm{Pic}_{X/S}$ is representable by a gp. scheme (Murre).

$\Downarrow$

Can make sense of subsheaf $\mathrm{Pic}^\circ_{X/k} \to$ represented by identity component of the gp. scheme $\mathrm{Pic}_{X/k}$.

$\Downarrow$

For a general base S, $\mathrm{Pic}^\circ_{X/S}$ is the subsheaf of sections which for every geometric point of S, map to the respective Pic°.

Let $S = \mathrm{spec}(\mathcal{O})$, $\mathcal{O}$ a dvr with fraction field K & residue field k. Suppose $\mathcal{X}/\mathcal{O}$ a curve ($\Rightarrow$ rel. dim $=1$). Let X_i be the irred. components of $\mathcal{X}_k$ with $d_i = \ell(\mathcal{O}_{\mathcal{X}, \eta_{X_i}})$.

Thm. (Raynaud)

Suppose a) $\mathcal{X}_K$ is smooth over K b) $\mathcal{X}$ is regular c) gcd of $d_i = 1$.

Let $\mathcal{J}$ be the Neron model of $\mathrm{Jac}(X_K)$ over $\mathcal{O}$ & $\mathcal{J}^\circ$ be the identity component. Then $\mathrm{Pic}^0_{\mathcal{X}/S}$ is infact representable by a smooth gp. scheme over $\mathcal{O}$ & coincides with $\mathcal{J}^\circ$.

$$\Downarrow$$

$$\mathcal{J}^\circ_k \xrightarrow{\sim} \mathrm{Pic}^0_{\mathcal{X}_k/k}.$$

Rmk.: In our situation $\mathcal{X}_k$ is reduced $\Rightarrow d_i's = 1$

Minimal regular model of $X_0(N)$:

We (as Snowden does) change some notations

- $N \longrightarrow p$ a prime
- We are going to ignore cusps (to be explained later)

$$\Downarrow$$

deal with $Y_0(p)$ instead of $X_0(p)$.

- Finally we will be working with the following stacks

a) $\mathcal{M}_0(p) \leftarrow$ stack parametrizing Elliptic curves and a subgp of order p.

$Y_0(p) = |\mathcal{M}_0(p)| \to$ coarse Moduli space, henceforth to be denoted by $M_0(p)$.

$\ell \neq p$ a prime > 2.

b) $\mathcal{M}_0(p, \ell) \leftarrow$ stack parametrizing Elliptic curves & a subgp of order p, together with a $\mathbb{F}_\ell$ basis for the ℓ-torsion points

$\mathcal{M}_0(p, \ell)$ is infact an Affine scheme and we shall denote it by $M_0(p, \ell)$.

c) Finally we have

$\mathcal{M}(\ell) \to$ Stack parametrizing Elliptic curves together with a basis for ℓ-torsion points.

$\mathcal{M}(\ell)$ is a scheme and we denote it by $M(\ell)$.

d) We have a presentation of $\mathcal{M}_0(p)$ as a DM stack.

$$M(p, \ell) \xrightarrow{\text{permuting a basis of } \ell\text{-torsion.}} \mathcal{M}_0(p) = [M(p, \ell)/G]$$

with $G = GL_2(\mathbb{F}_\ell) \to$ automorphism gp. of (split) ℓ-torsion points on an Elliptic curve over a field.

$\Rightarrow M_0(p) = M(p, \ell)^{G} \leftarrow$ schematic quotient obtained by taking G invariants of the coordinate ring.

$\underline{\ast}$: Note that $\# G = (\ell^2 - 1)(\ell^2 - \ell) \not\equiv 0 \pmod p$

Thus taking invariants behaves well even in char p.

e) Finally if in all the above statements we replace Elliptic curves with 'generalized Elliptic curves' we get models over $\mathbb{Z}$.

Fact: a) $\mathcal{M}(\ell)$ is smooth over $\mathbb{Z}[\frac{1}{\ell}]$

b) $\mathcal{M}_0(p,\ell)$ is regular and flat over $\mathbb{Z}[\frac{1}{\ell}]$. Moreover it is smooth over $\mathbb{Z}[\frac{1}{p\ell}]$.

AIM: To study the singularities in the special fiber of the minimal regular model of $M_0(p)$ by studying those of $M_0(p,\ell)_{\mathbb{F}_p}$.

Prop^n: The scheme $M_0(p,\ell)_{\mathbb{F}_p}$ (curve over $\mathbb{F}_p$) is CM & reduced. It is smooth (over $\mathbb{F}_p$) away from the s.s. points. Each s.s. point is an ordinary node.

Pf. Let $M_0(p,\ell) = \mathrm{Spec}(A)$. A is regular & flat over $\mathbb{Z}[\frac{1}{\ell}]$. (no-torsion $\Rightarrow p$ is a nzd.)

Thus $B = \frac{A}{pA}$ is CM (quotient of a regular ring by a nzd)

In the rest of the proof we work over $\overline{\mathbb{F}}_p$.

(1) We have morphisms

$$i : \mathcal{M}(\ell)_{\overline{\mathbb{F}}_p} \to M_0(p,\ell)_{\overline{\mathbb{F}}_p}$$
$$[E,(P,Q)] \mapsto [E, \ker F, (P,Q)]$$

$$j : M(\ell)_{\overline{\mathbb{F}}_p} \to M_0(p,\ell)_{\overline{\mathbb{F}}_p}$$
$$[E,(P,Q)] \mapsto [E^{(p)}, \ker(V), (P,Q)]$$

$$f : \mathcal{M}_0(p,\ell)_{\overline{\mathbb{F}}_p} \to \mathcal{M}(\ell)_{\overline{\mathbb{F}}_p}$$ source of p-isogeny

$$g : \mathcal{M}_0(p,\ell)_{\overline{\mathbb{F}}_p} \to \mathcal{M}(\ell)_{\overline{\mathbb{F}}_p}$$ target of p-isogeny.

Clearly by construction $f \circ i = 1_{\mathcal{M}(\ell)}$, $g \circ j = 1_{\mathcal{M}(\ell)}$

$\Rightarrow i$ & j are closed immersions.

(2) Since everything in sight is a curve & $M(\ell)_{\overline{\mathbb{F}}_p}$ is smooth over $\overline{\mathbb{F}}_p$, i & j must infact be just immersion of components.

(3) Finally note that $\mathcal{M}_0(p,\ell)^{ss}_{\overline{\mathbb{F}}_p} \leftarrow$ closed subset of $\mathcal{M}_0(p,\ell)_{\overline{\mathbb{F}}_p}$.

(given by vanishing of a morphism of vector bundles on $\mathcal{M}_0(p,\ell)$)

$$\mathcal{M}_0(p,\ell)^{ord}_{\overline{\mathbb{F}}_p} = \mathcal{M}_0(p,\ell)_{\overline{\mathbb{F}}_p} \setminus \mathcal{M}_0(p,\ell)^{ss}_{\overline{\mathbb{F}}_p}$$

$$= i(\mathcal{M}^{ord}(\ell)) \coprod j(\mathcal{M}^{ord}(\ell))$$

This is thus smooth over $\overline{\mathbb{F}}_p$.

$\Downarrow$

$\mathcal{M}_0(p,\ell)_{\overline{\mathbb{F}}_p}$ is generically reduced, CM & one-dimensional.

$\Downarrow$

$\mathcal{M}_0(p,\ell)_{\overline{\mathbb{F}}_p}$ is reduced ($\Rightarrow \mathcal{M}_0(p,\ell)$ is reduced)

[Proof: Let A be a generically reduced CM ring. Need to show A is reduced. Clearly may assume A is local. Suppose $f \in A$ is nilpotent. Then $m_A f = 0$, since A is generically reduced.

$\not\Rightarrow$ every element of m_A is a 0-divisor a contradiction]

4) $M_0(\ell, P)_{\overline{\mathbb{F}}_p}$ is defined over $\mathbb{F}_p$. Thus there is a Frobenius $F: M_0(\ell,P)_{\overline{\mathbb{F}}_p} \to M_0(\ell,P)_{\overline{\mathbb{F}}_p}$ whose fixed points are $\mathbb{F}_p$-valued points

points are $\mathbb{F}_{p}$-valued points.

i) By construction it is immediate that $F \circ i = j$, $F \circ j = i$
In particular F^2 preserves $i(\mathcal{M}(\ell)_{\overline{\mathbb{F}}_p})$ & their intersection
$j(\mathcal{M}(\ell)_{\overline{\mathbb{F}}_p})$

ii) It also follows that $f \circ j = F = g \circ i$

Rmk.: 4) $\Rightarrow$ s.s elliptic curves over $\overline{\mathbb{F}}_p$ are infact defined over $\mathbb{F}_{p^2}$.

5) Now let $x \in \mathcal{M}(\ell)$ be a s.s. point. Let $y = i(x) = j(x) \in \mathcal{M}_0(p,\ell)$.
We have a map

$$\begin{array}{ccl} \mathcal{M}(\ell)_{\overline{\mathbb{F}}_p} \amalg \mathcal{M}(\ell)_{\overline{\mathbb{F}}_p} & \rightarrow & \text{smooth curves over } \overline{\mathbb{F}}_p \\ \big\downarrow{\scriptstyle i \amalg j} & & \text{normalization} \\ \mathcal{M}_0(\ell,p)_{\overline{\mathbb{F}}_p} & \rightarrow & \text{reduced, generically smooth curve} / \overline{\mathbb{F}}_p . \end{array}$$

Let A_y be the completed local ring at y.
" C_x " " " " " x.

Then $\underbrace{(i \amalg j)^*}_{= \alpha} : A_y \to C_x \times C_x$ is injective since A_y is reduced.

Let $t \in C_x$ be an uniformizer.

Let $u = f^*(t) - g^*(t^p)$ $\Rightarrow$ $\alpha(u) = (i \amalg j)^* f^*(t) - (i \amalg j)^* g^*(t^p)$

$$= \big(t,\ j^* \circ f^*(t)\big) - \big(i^* g^*(t^p),\ t^p\big)$$

$$= (t, t^p) - (t^{p^2}, t^p)$$

$$= (t - t^{p^2}, 0) = \big(\underbrace{t(1-t^{p^2-1})}_{\text{also an unifor}}, 0\big)$$

miter.

Similarly if $v = g^*(t) - f^*(t^p)$

$$\Rightarrow a(v) = \left(0,\ t - t^{p^2}\right) = \left(0,\ t(1 - t^{p^2-1})\right)$$

Upshot: any element of $C_x \times C_x$ is a power series in $a(u)$ & $a(v)$

$\Downarrow$

if $z \in m_{A,y}$ then $a(z) = F(a(u), a(v))$ for some power series F.

Since a is injective. This $\Rightarrow \bar{\mathbb{F}}_p[[u,v]] \twoheadrightarrow A_y$

Since $a(uv) = 0 \Rightarrow$

$$\begin{tikzcd} \bar{\mathbb{F}}_p[[u,v]] \arrow[d, two heads] \arrow[dr, two heads] & \\ \dfrac{\bar{\mathbb{F}}_p[[u,v]]}{uv} \arrow[r, two heads, "\varphi"'] & A_y \end{tikzcd}$$

Both of these are reduced local rings with two components, so φ is an iso.

$\Downarrow$

Any s.s. point has a nodal singularity

$\square$

Now we carry out the proof an integral version of the above proposition.

Prop^n: The scheme $M_0(p, \ell)$ is smooth over $\mathbb{Z}[\frac{1}{\ell}]$ away from the s.s points in char p. At the s.s points the strictly complete local ring is $\xrightarrow{\sim} \dfrac{W(\bar{\mathbb{F}}_p)[[u,v]]}{(uv - p)}$ $\leftarrow$ regular.

Pf. Smoothness can be checked fibre wise so the result on smoothness follows from last prop^n.

1) Let $R \to$ strictly complete local ring at x a s.s point in char p.

$R \Rightarrow$ flat $/\mathbb{Z}[\frac{1}{\ell}]$, $\dim(R) = 2$

$$\frac{R}{pR} \xrightarrow{\sim} \frac{\overline{\mathbb{F}}_p[[u,v]]}{uv}$$

$$\begin{tikzcd}
W(\overline{\mathbb{F}}_p)[[u,v]] \arrow[r, two heads] \arrow[d, dotted, "\phi"', "\text{choose lifts of } u \,\&\, v"] & \overline{\mathbb{F}}_p[[u,v]] \arrow[d, two heads] \\
R \arrow[r, two heads] & \frac{R}{pR}
\end{tikzcd}$$

local rings

$\Downarrow$

ϕ is surjective since it is so mod p by Nakayama.

Moreover $uv \in (p) \Rightarrow uv = pw \Rightarrow \underbrace{\frac{W(\overline{\mathbb{F}}_p)[[u,v]]}{(uv - pw)}}_{\text{completed local ring of a scheme flat over } W(\overline{\mathbb{F}}_p)} = R' \overset{\phi}{\twoheadrightarrow} R.$

$\Rightarrow R'/W(\overline{\mathbb{F}}_p)$ is flat

Moreover ϕ mod p is an iso $\Rightarrow \ker(\phi)$ is p-torsion & hence trivial by flatness.

2) Now we prove that w has to be an unit. Indeed $\dim(R) = 2$

$\Rightarrow$ By regularity $\dim_{\overline{\mathbb{F}}_p} \frac{m}{m^2} = 2$. But $m = (u, v, p)$

& $m^2 = (uv, up, vp) = (pw, up, vp)$

Thus $\frac{m}{m^2}$ = Vecto rspace spanned by u, v, p mod $p\omega$

If ω is not an unit $p\omega \in m^2$ & $\dim_{\mathbb{F}_p} \frac{m}{m^2} \geq 3$ contradicting regularity.

$\Downarrow$

Upto rescaling $R \cong \frac{W[[u,v]]}{(uv-p)}$

Structure of $M_0(p)$

- Recall that $M_0(p) = M_0(p, \ell)/G$. So we can use the above Propositions to understand the local ring at s.s points in char. p.

- As observed before reduction mod p commutes with taking quotients by G since $p \nmid |G|$.

$\Downarrow$

$$M_0(p)_{\overline{\mathbb{F}_p}} = M_0(p, \ell)_{\overline{\mathbb{F}_p}} / G.$$

- To understand the local ring of $M_0(p)$ at a s.s point we need to understand the invariants under the action of stabilizers on a point above on $M_0(p, \ell)$.

(see [DR] Chap VI Section 6)

Thm. Let x be a char. p point of $M_0(p)$ & R be the strict complete local ring at x.

a) If x is not ss $M_0(p)$ is smooth at x.

b) If x is s.s & $j(x) \neq 0, 1728$ $M_0(p)$ is regular at x &
$R = W[[x,y]] / (xy - p)$

c) If x is s.s & $j(x) = 1728$, $R = W[[x,y]] / (xy - p^2)$ $\Big) \rightarrow$ not regular

d) " " " " & $j(x) = 0$, $R = W[[x,y]] / (xy - p^3)$

Rmk. With further analysis one can show that cuspidal points of $\overline{M}_0(p)$ are smooth

Minimal regular model of $\overline{M}_0(p)$

$\overline{M}_0(p)$ is a flat proper model of its generic fiber which is regular at all points except those in char p & have j invariants 0 or 1728. To make this regular one has to blow up these points (once for 1728, twice for 0) which introduces $\mathbb{P}^1$'s on the special fibers.

$\Downarrow$

Reduction mod p of the minimal regular model is a reduced curve , all of whose irred. components are a bunch of $\mathbb{P}^1_s$' intersecting transversally.

Finally an easy computation tells you that such a curve has Pic^0 a torus.

$\Downarrow$ $J_0(p)$ mod p has toric reduction.